CRACKING RESISTANCE OF A STAR CONTOUR

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When a plate is penetrated by a star-shaped missile [1, 2], pierced holes of exotic shape are formed. The hole either reproduces the edge of the missile or circumscribes it. The angular points in the missile correspond either to rounded parts in the hole or to cracks emerging at the edge of it.

To simplify the treatment, we examined the first basic topic in the theory of elasticity [3] in relation to a star-shaped hole and missile, which corresponds to the real physical picture [4].

Here we consider cases where the star-shaped hole has a curvilinear boundary. The contour curves enclosing the boundary have common tangents at the ends of the rays, i.e., they degenerate into sections. We derived the dependence of the stress-intensity coefficient for the ends of the sections on the number of rays and the ratio of the maximum radius in the star to the minimum. The behavior of a sufficiently long crack at the end of a ray is examined by reference to a plane with an angular slot and the line of section continuing it. We consider a star with rounded corners, and we derive the stress-concentration coefficient for those corners in relation to the above parameters and the radius of curvature.

1. The stress function can be put as the following contour integral [5] as a solution to the biharmonic equation:

$$U^0(x, y) = \frac{1}{2\pi}\int_L [\rho_1(x, y, t) f_1(t) + \rho_2(x, y, t) f_2(t)] \ln r^2(x, y, t)\, dt, \tag{1.1}$$

$$\rho_1 = a(t)[x - x(t)] + b(t)[y - y(t)],\ \rho_2 = -b(t)[x - x(t)] + a(t)[y - y(t)],$$

$$r^2 = \rho_1^2 + \rho_2^2,$$

where $x(t)$, $y(t)$ is a point on the contour L corresponding to a given value of the parameter t, and $a(t)$, $b(t)$ represent unit vector for the tangent to L at that point. We introduce additional terms that incorporate the homogeneous state of stress in the plane and derive the stress function in the form

$$U(x, y) = U^0(x, y) + 0{,}5(\sigma_1 y^2 + \sigma_2 x^2). \tag{1.2}$$

If L is formed by n smooth curves, $L = \bigcup_{k=0}^{n-1} L_k$, then

$$U^0(x, y) = \sum_{k=0}^{n-1} \frac{1}{2\pi}\int_{L_k} (\rho_{1k} f_{1k} + \rho_{2k} f_{2k}) \ln r_k^2 dt.$$

Here the kernels ρ_{1k}, ρ_{2k} are expressed in terms of the functions $x_k(t)$, $y_k(t)$ defining L_k in accordance with (1.1). With cyclic symmetry in the disposition of the L_k curves and in the distribution of the external forces, we have $f_{jk}(t) = f_j(t)$, $j = 1, 2$, $k = 0, 1, \ldots, n - 1$.

The normal and tangential stresses in the curvilinear coordinate system ξ, η are defined by

$$\sigma_\eta = \sigma_x \sin^2\alpha + \sigma_y \cos^2\alpha + \tau_{xy} \sin 2\alpha,$$
$$\tau_{\xi\eta} = 0.5(\sigma_y - \sigma_x) \sin 2\alpha + \tau_{xy} \cos 2\alpha, \tag{1.3}$$

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where $\sigma_x = \partial^2U/\partial y^2$; $\sigma_y = \partial^2U/\partial x^2$; $\tau_{xy} = -\partial^2U/\partial x\partial y$, and α is the angle between the tangent to the line η = const in the direction of increase in t and the x axis. On L

$$\exp(2i\alpha) = c + is = [x'(t) + iy'(t)]/[x'(t) - iy'(t)].$$

We apply to (1.2) boundary conditions on L in accordance with (1.3):

$$\sigma_\eta = p_1(t),\ \tau_{\xi\eta} = p_2(t),$$

which gives us the singular integral equation

$$\frac{1}{\pi}\int_L G(t, \tau) f(\tau)\, d\tau = p(t), \tag{1.4}$$

where p(t) is the external-force vector, f(t) is the unknown-function vector, and $G(t, \tau)$ is a square matrix of order 2n whose diagonal elements contain a singularity of Cauchy type.

We consider a plane having a hole in the form of a curvilinear star, whose contour is tangential to the rays $\varphi = \varphi_k = 2k\pi/n,\ k = 0, 1, \ldots, n-1$ and is defined by the equation $r = (\varphi)$ given in polar coordinates φ and r. By virtue of the cyclic symmetry, we put $r(\varphi + 2\pi/n) = r(\varphi),\ p(\varphi + 2\pi/n) = p(\varphi)$.

For $r(-\varphi) = r(\varphi),\ p_1(-\varphi) = p_1(\varphi),\ p_2(\varphi) = 0$, vector equation (1.4) leads to the singular integral equation

$$\frac{1}{\pi}\int_0^{2\pi/n} f_1(\tau)\left[\frac{G_0(t, \tau)}{t - \tau} + \sum_{k=1}^{n-1} G_k(t, \tau)\right] d\tau = p_1(t) \tag{1.5}$$

for the unique unknown function $f_1(t)$, where

$$\begin{aligned}
&G_k(t, \tau) = (a_kU_k + b_kV_k)/D_k + (a_kV_k - b_kU_k)\left[2cU_kV_k - s\left(U_k^2 - V_k^2\right)\right]/D_k^2,\\
&x_k(t) = r(t)\cos(\varphi_k - t),\ y_k(t) = r(t)\sin(\varphi_k - t),\\
&D_k = U_k^2 + V_k^2,\quad a_k = x_k'(\tau),\quad b_k = y_k'(\tau),\\
&U_k = [x_0(t) - x_k(\tau)]\gamma_k(t, \tau),\ V_k = [y_0(t) - y_k(\tau)]\gamma_k(t, \tau),\\
&\gamma_0(t, \tau) = 1,\ \gamma_k(t, \tau) = (t - \tau)^{-1},\ k = 1, \ldots, n-1.
\end{aligned}$$

To (1.5) we add the condition that the displacements are unique:

$$\int_L f(t)\, dt = 0. \tag{1.6}$$

The method of [6] is used to reduce (1.5) and (1.6) to a system of linear algebraic equations:

$$\sum_{i=1}^{N} A_{ij} f_j = p_j. \tag{1.7}$$

For this purpose we put

$$t = \frac{\pi}{n}(1 + \cos\omega),\quad \tau = \frac{\pi}{n}(1 + \cos\theta),$$

$$f(\tau) = \frac{1}{N\cos\theta}\sum_{j=1}^{N}(-1)^{j+1} f_j \frac{\cos N\theta\sin\theta_j}{\cos\theta - \cos\theta_j}.$$

Expressions have been given [6] for the coefficients A_{ij} in (1.7).

The stress distribution near the end $r = r_0$ of the ray $\varphi = \varphi_k$ takes the form $\sigma_\theta = K(r - r_0)^{-0.5}$ apart from infinitely small quantities, where the parameter K, the stress-intensity coefficient, is proportional to the following [5]

$$\varkappa = \frac{1}{2N}\sum_{j=1}^{N}(-1)^j f_j \operatorname{ctg}\frac{\theta_j}{2}. \tag{1.8}$$

In the calculations we took N as 17 and 21, with the equation for the contour

$$r(\varphi) = r_1 - \frac{n}{\pi}(r_1 - r_2)\sqrt{\varphi\left(\frac{2\pi}{n} - \varphi\right)}, \quad 0 \leqslant \varphi \leqslant \frac{2\pi}{n},$$

and the area of the star

$$S = \frac{\pi}{6}\left[(10 - 3\pi) r_1^2 + 4r_2^2 + (3\pi - 8) r_1 r_2\right] = \pi,$$

where r_1 is the minimum radius and r_2 is the maximum one. We varied the parameters n and $\varepsilon = r_1/r_2$.

The calculations show (Fig. 1) that $\varkappa$ takes its largest values for $\varepsilon > 2.25$ with $n = 10$, for $1.75 < \varepsilon < 2.25$ with $n = 9$, and for $1.5 < \varepsilon < 1.75$ with $n = 8$. The differences in these values are slight.

2. We used the integral representation for the general solution to (1.1) to examine a plane with an angular slot and a crack at its vertex directed along the bisector of the angle. The contour L is formed by the right-hand boundary of the angle $L_1\{x = at,\ y = bt,\ 0 < t < \infty\}$, the left-hand boundary $L_2\{x = at,\ y = -bt,\ -\infty < t < 0\}$, and the line of section $L_3\{x = 0,\ y = t,\ -l < t < 0\}$, where $a = \cos\alpha$, $b = \sin\alpha$, and α is the inclination of the right boundary to the x axis.

We assume that the external forces are applied symmetrically with respect to the axis of the line of section to get a system of singular integral equations

$$\frac{1}{\pi}\int_{L_i}\frac{f_i(\tau)\,d\tau}{t - \tau} + \sum_{j=1}^{3}\frac{1}{\pi}\int_{L_j} f_j(\tau)\, G_{ij}(t, \tau)\, d\tau = p_i(t), \quad i = 1, 2, 3. \tag{2.1}$$

The kernels G_{ij} are not written out. If L_3 is absent and the variation in t is restricted, (2.1) takes the form obtained on examining two cracks at an angle to one another [7]. In place of (1.6) we add the condition $f_j(0) = 0$, which provides for bounded stresses at the vertex of the cut angle. Equation (2.1) was also solved numerically by the method of [6]. In the calculations it was assumed that the external forces were constant, orthogonal to the axis of the line of section, and act for $t \leqslant 1$, i.e.,

$$p_1 = b^2 h(t), \quad p_2 = -abh(t), \quad p_3 = 1,$$

where $h(t) = 1$ for $t \leqslant 1$ and $h(t) = 0$ for $t > 1$.

Figures 2 and 3 show calculations on the stress intensity coefficient at the end of the crack $K = \varkappa\sqrt{l/2}$, where $\varkappa$ is defined by (1.8). Curves 1-5 (Fig. 2) correspond to angles in the cutout of 150, 120, 90, 60, and 30°. K increases monotonically with the crack length. The variation in K as the angle of the cutout decreases (Fig. 3) is not monotone: near $\alpha \sim 35$-$45°$ (cutout angle about 90°), the curves have maxima, i.e., the plane is most damaged.

3. We estimate the stresses when the corners in the star-shaped hole and the missile are rounded, using the method of [3].

Consider a star-shaped figure in the plane of the complex variable z bounded by a kinked line with its vertices at the points

$$A_k'\left(z_k' = R\exp(2\pi ik/n)\right), \quad A_k''\left(z_k'' = r\exp(\pi i(2k + 1)/n)\right),$$
$$k = 0, 1, 2, \ldots, n - 1,$$

with the number of rays for the star $n = 2, 3, \ldots$. We take the area of the star as $\pi\rho_0^2$ (ρ_0 = const) to get a relation between the parameters

$$\alpha = \frac{2}{\pi}\operatorname{arcctg}\left\{\frac{\pi}{n}\lambda^2\sin^{-2}\frac{\pi}{n} - \operatorname{ctg}\frac{\pi}{n}\right\}, \quad \beta = \alpha + \frac{2}{n}, \quad \lambda = \frac{\rho_0}{r},$$

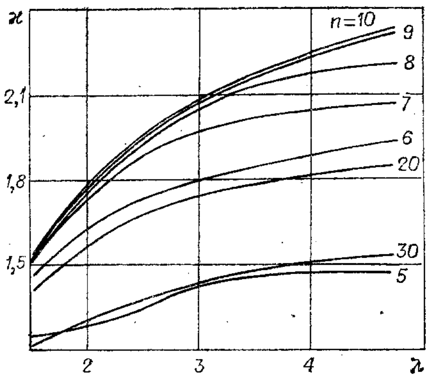

Fig. 1

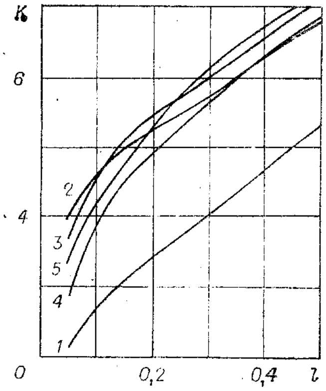

Fig. 2

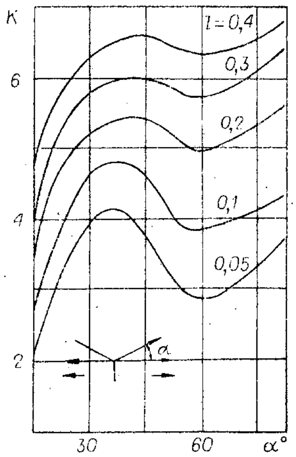

Fig. 3

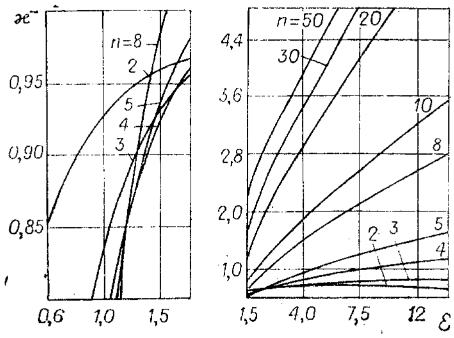

Fig. 4

TABLE 1

M	20		40		60	
	$\varkappa^-$	ρ^*	$\varkappa^-$	ρ^*	$\varkappa^-$	ρ^*
$n=3$	0,863	0,423	0,912	0,196	0,932	0,127
$n=6$	0,914	0,161	0,949	0,094	0,951	0,069

$$R = r \sin\frac{\beta\pi}{2}\sin^{-1}\frac{\alpha\pi}{2}.$$

We denote the region within the star by D_z^+ and that outside it by D_z^-, while the region in the ζ plane within unit circle $\{|\zeta| = 1\}$ is denoted by D_ζ^+ and outside it by D_ζ^-. The conformal mapping of D_z^+ into D_ζ^+ has the form of a Christoffel-Schwartz integral:

$$z = \omega^+(\zeta) = c_0^+ \int_0^\zeta (1+\xi^n)^{1-\beta}(1-\xi^n)^{\alpha-1}\,d\xi, \quad \zeta = \rho e^{i\theta},$$

where c_0^+ is derived from the condition $\omega^+(1) = R$. Similarly,

$$z = \omega^-(\zeta) = c_0^- \int_0^{1/\zeta} (1+\xi^n)^{\beta-1}(1-\xi^n)^{1-\alpha}\,d\xi + R,$$

$$\omega^-(r\exp(\pi i/n)) = \exp(\pi i/n)$$

perform the conformal mapping of D_z^- into D_ζ^-.

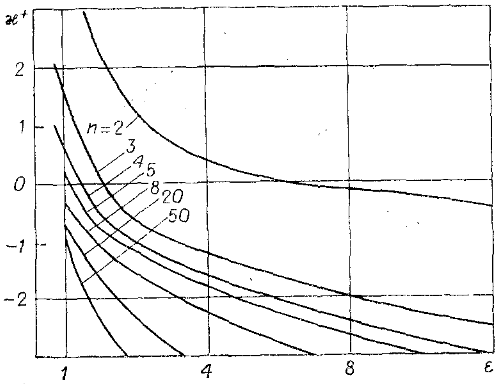

Fig. 5

Let the normal force $\sigma_n = p(t)$ be given at the edge of the star. The first boundary-value problem for the Kolosov–Muskhelishvili functions $\varphi^*(z)$, $\psi^*(z)$ in the regions D_z^+ (internal problem) and D_z^- (external problem) is written as

$$\varphi_\pm^*(t) + t\overline{\varphi_\pm^{*\prime}(t)} + \overline{\psi_\pm^*(t)} = \pm q(t)\exp(-\alpha\pi i/2), \quad q(t) = \int\limits_R^t p(\tau)\,d\tau$$

or after conformal transformation

$$\varphi_\pm(\sigma) + \frac{\omega^\pm(\sigma)}{\overline{\omega^{\pm\prime}(\sigma)}}\overline{\varphi_\pm'(\sigma)} + \overline{\psi_\pm(\sigma)} = \pm q(\omega^\pm(\sigma))\exp(-\alpha\pi i/2), \tag{3.1}$$
$$\varphi(\zeta) = \varphi^*(\omega,(\zeta)),\ \psi(\zeta) = \psi(\omega(\zeta)).$$

The method of solving (3.1) approximately is as follows. The expansions of $\omega^+(\zeta)$ in the region of $\zeta = 0$ and of $\omega^-(\zeta)$ in the region $\zeta = \infty$ take the form

$$\omega^\pm(\zeta) = \sum_{k=0}^\infty c_k^\pm \zeta^{1\pm kn}, \tag{3.2}$$

where

$$c_k^\pm = \frac{c_0^\pm}{1 \pm kn}\sum_{j=0}^k a_j^\pm b_{k-j}^\pm, \quad a_0^\pm = b_0^\pm = 1,$$
$$a_k^+ = a_{k-1}^+\frac{k-\alpha}{k}, \quad b_k^+ = b_{k-1}^+\frac{2-k-\beta}{k}, \quad a_k^- = a_{k-1}^-\frac{k+\alpha-2}{k}, \quad b_k^- = b_{k-1}^-\frac{\beta-k}{k}.$$

By virtue of this, we seek the functions $\varphi(\zeta)$, $\psi(\zeta)$ in the form

$$\varphi_\pm(\zeta) = \pm p\exp(-\alpha\pi i/2)\sum_{k=0}^\infty \alpha_k\zeta^{1\pm kn}, \tag{3.3}$$

$$\psi_\pm(\zeta) = \pm p\exp(-\alpha\pi i/2)\sum_{k=0}^\infty \beta_k\zeta^{-1\pm kn}.$$

We substitute (3.2) and (3.3) into (3.1) and compare the coefficients to identical powers of ζ to get a system of algebraic equations for the coefficients α_k and β_k. A finite number of terms was retained in the calculations on the expansions of (3.2) and (3.3), and this assumption, which enables one to reduce the problem to a finite system of equations and to derive finite values for the stress-concentration coefficient at the nodal points, is equivalent to rounding the corners at the vertices of the star. The rounding is characterized by the radius of curvature.

One can evaluate the effects of the number of rays and the elongation of them on the state of stress in the internal or external region by means of the stress-concentration coefficient $\sigma_\theta^\pm = 4\ \mathrm{Re}\ \varphi_\pm^{*\prime}(z) - p$ as calculated at the point ($\rho = 1$, $\theta = \pi/n$) in the first case and at the point ($\rho = 1$, $\theta = 0$) in the second. In the calculations, n was varied from 2 to 50 and ε from 0.6 to 15, this being the ratio of the maximum radius of the star to the minimum.

Figures 4 and 5 give graphs for $\varkappa^- = \sigma_\theta^-/p$ and $\varkappa^+ = \sigma_\theta^+/p$ as ε varies for various n. As ε increases for each value of n, $\varkappa^-$ increases monotonically. However, $\varkappa^-$ increases if $\varepsilon < 1.6$ for a given ε and as n increases. For example, with $\varepsilon = 1.1$, $\varkappa^-$ takes its least value when $n = 7$ ($\alpha = 0.68$, $\beta = 0.96$) and its largest when $n = 2$ ($\alpha = 1/3$, $\beta = 4/3$). If $\varkappa^+$ is positive, the missile has the form of a convex figure.

One can optimize the form of the missile subject to constraints imposed by strength requirements on $\varkappa^+$ by choosing n and ε such that $\varkappa^-$ takes its maximum value. A multirayed form is preferable for a high-strength missile.

The effects of the rounding at the vertex of a ray on the stress-concentration coefficient for the star-shaped hole are given in the table when one takes a finite number of terms in (3.2) and (3.3). We give the values of $\varkappa^-$ and the radius of curvature ρ^* at the vertex of a rounded corner for $\varepsilon = 1.6$ and $n = 3$ and 6 on retaining $M = 20$, 40, or 60 terms.

A similar picture is observed for other values of n and ε.

The experimental studies on the strength characteristics of star-shaped missiles and holes [1, 8] lead us to recommend results obtained for curvilinear stars with sharp corners and corners with cracks as in sections 1 and 2 on impact with brittle materials. The results of section 3 in that case may be incorrect. Their use is desirable for convex polyhedra and stars with blunt corners.

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